We analyse a variety of ways that comparing dice values can be used to simulate battles in games, measuring the ‘win bias’, ‘tie percentage’, and ‘closeness’ of each variant, to provide game designers with quantitative measurements of how small rule changes can significantly affect game balance. Closeness, a metric we introduce, is related to the inverse of the second moment, and measures how close the final scores are expected to be. We vary the number of dice, number of sides, rolling dice sorted or unsorted, biasing win rates by using mixed dice and different number of dice, allowing ties, re-rolling ties, and breaking ties in favour of one player.

1 Introduction

Dice are a popular source of randomness in games. We examine the use of dice to simulate combat and other contests. While some games have deterministic rules for exactly how a battle will resolve, many games add some randomness, so that it is uncertain exactly who will win a battle. In games such as Risk [1], two players roll dice at the same time, and then compare their values, with the higher value eliminating the opponent’s unit. Other games use a hit-based system, such as Axis and Allies [2], where a die roll of a strength value or less is a successful hit, with stronger units simulated by larger strength values and larger armies rolling more dice. In both games, stronger forces are more likely to win the battle, but lucky or unlucky rolls can result in one player performing far better, making a wide difference in scores.

Given a very large number of games played between players, unlucky and lucky rolls will balance out such that the player who has better strategy will probably end up winning; however, people might not play the same game enough times for the probabilities to even out. Instead, they play a much smaller, finite number of rolls spread across one session, or perhaps a couple of play sessions. The gambler’s fallacy is the common belief that dice act with local representativeness: even a small number of dice rolls should be very close to the expected probabilities [3, 4, 5]. Therefore, it can often be frustrating when rolling poorly against an opponent: players often blame the dice, or themselves, for bad rolls, even though logic and reason dictate that everyone has the same skill at randomly rolling dice. Game designers may want to avoid or reduce this kind of negative player experience in their games.

Although there are thousands of games based on dice (the BoardGameGeek online database lists over 7,000 entries for dice games [6] and hundreds of games are described in detail in [6] [7]), we specifically examine games where players roll and compare the individual dice outcomes, as in Figure 1. Each player’s dice are sorted in decreasing order and then paired up. Whichever player rolled a higher value in a pair wins a point. The points are summed, and whoever has more points wins the battle.

We use the term battle to denote any event resolved randomly within a larger game. The word is normally used to refer to combat, but our analysis can be used any time players compare dice outcomes in a contest. In this paper, we will use the terms battle and game interchangeably.
We examine different variants and show how different factors affect the distribution of scores and other metrics which are helpful for evaluating a game. By adjusting the dice mechanics, a designer can influence the:

1. expected **closeness** of battle outcomes,
2. **win bias** in favour of one of the players, and
3. **tie percentage**, i.e. the fraction of battles that end in a tie.

The variants we examine include: different numbers of dice; different numbers of sides per die; different ways to sort the dice; and various ways to break ties.

Dice have come with various numbers of sides for millennia [8]: some of the oldest dice, dating back to at least 3,500BC, were bones with four flat sides and two rounded ones. Eventually, 6-sided dice were created by polishing down the rounded sides. The dot patterns we see on today’s 6-sided dice also come from antiquity. Ancient dice also come in the form of sticks with four long sides for Pachisi or two long sides for Senet [9]. The common dice in use for modern games are 4-, 6-, 8-, 10-, 12- and 20-sided, but other variants exist. In this paper, we use the notation $n$-sided dice with $k$ sides each, e.g. 5d6 means a roll of five 6-sided dice.

By understanding how rules and randomness affect closeness, a designer can then choose the appropriate combination to try to achieve their desired game experience. A designer may prefer for their game to be highly unpredictable with large swings, intentionally increasing the risk for players to commit their limited resources. In addition, randomness can make a game appear to be more balanced because the weaker player can occasionally win against the stronger player [10]. Large swings may be more emotional and chaotic, and the ‘struggle to master uncertainty’ can be considered ‘central to the appeal of games’ [11]. Or, a designer may prefer for each battle to be close, to limit the feelings of one side dominating the other in what might be experienced as unfair or unbalanced, in a trait known as inequity aversion [12][13]. Similarly, a designer may prefer to allow ties (simulating evenly matched battles), or wish to eliminate the opportunity for ties (forcing one side to win). Finally, a designer might want to vary the rules between each battle within a game, to represent changing strengths and weaknesses of the players or to provide aid to the losing player. A designer can adjust randomness to encourage situations appropriate for their game.

For most sections in this paper, we calculate the exact probabilities for each outcome by iterating over all possible rolls, tabulating the final score difference. Because each outcome is independent, we can parallelise the experiments across multiple processors to speed up the calculations (details about the calculations are given in the Appendix). There are other methods one could use to computationally evaluate the odds, such as a dice probability language such as AnyDice [10] or Troll [14], or by using Monte Carlo simulation (we use simulation when examining re-rolls in Section 9). Writing the analytical probabilities becomes difficult for more complex games, and we feel that presenting such equations is less useful for most game designers.

## 2 Metrics for Dice Games

Quantitative metrics have been used to computationally analyse outcome uncertainty in games, typically for the purpose of generating novel games [15][16]. Here we focus only on metrics that examine the final scores of the dice battle; we do not evaluate anything about how scores evolve during the battle itself (which we believe would be essential for more complicated games). But for simple dice battles, which are a component of a longer game, we can just focus on the end results.

**Win bias** and **tie probability**, are similar to those used in previous work, but one of our metrics, **closeness**, is something we have not seen used before in game analysis. We now define these metrics precisely.

### 2.1 Score Difference

Let $s_A$ be the final score of a battle for Player A, and $s_B$ be the final score for Player B. The **battle score** or **score difference**, $d$, is given by $d = s_A - s_B$. If we iterate over all the possible ways that the dice can be rolled, counting the number of times that each score difference $d$ occurs, we can derive a **score difference probability distribution**, $D(d)$. This describes the probability of achieving a score difference of $d$ in the battle. We calculate $D(d)$ by first counting every resulting score difference in a histogram-like data structure, and then dividing each bin by the total sum of all bins.

### 2.2 Win Percentage

The **win percentage** is the percent probability of Player A winning a battle. This can be calculated by summing the probabilities where the score difference is positive and is therefore a win for

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2 [http://anydice.com](http://anydice.com)
Player A. This is calculated as $100 \sum_{d<0} D(d)$ and will be between 0 and 100. Loss percentage is the percent probability of Player A losing a battle, and is calculated as $100 \sum_{d<0} D(d)$, also between 0 and 100.

### 2.3 Win Bias

The difference of the win and loss percentages gives the win bias:

$$\text{win bias} = 100 \left( 100 - \sum_{d>0} D(d) - \sum_{d<0} D(d) \right)$$

This will range from -100 to 100. Games with a win bias of 0 are balanced, with no preference of Player A over Player B. If the win bias is > 0, then Player A is favoured; if < 0, then Player B is favoured. This metric is similar to the balance metric in [15], but here we include the effect of ties and are concerned with the direction of the bias. A non-zero win bias is often desired, for example, when simulating that one player is in a stronger situation than the other.

### 2.4 Tie Percentage

*Tie percentage* indicates the percent probability of the battle ending in a tie, which we define as:

$$\text{tie percentage} = 100 \left( D(0) \right)$$

Some designers may want a possibility of ties, while others may not. This metric is analogous to *drawishness* in [15].

### 2.5 Closeness

Finally, we present *closeness*, our new metric which measures how much the final score values centre around a tied game. Games that often end within 1 point should have higher closeness than games that often end with a score difference of 5 or -5. The related statistical term *precision* is defined as the inverse of the variance about the mean. For closeness, we define this as the square root of the inverse of variance in the score difference distribution about the tie value $d = 0$:

$$\text{closeness} = \frac{1}{\sqrt{\sum d^2 D(d)}}$$

To explain this, we look at the denominator, which is similar to the *standard deviation*, i.e. the square root of the *variance*. However, we do not want this to be centred about the mean, as in the typical formulation. A game that always ends tied 0-0 would have a variance of 0, but so would a game that always ends in a 5-0 win, because the outcome (mean) is always the same. Yet 5-0 is certainly not a close score. Thus, we centre the second moment around 0 since close games are those with final score differences close to 0. Finally, we take the inverse because we want the metric to increase as the scores become closer and to decrease as the scores become further apart.

This formulation captures the familiar notion of a ‘close game’ and hence has some intuitive meaning. Closeness approaching 0 means that the final score differences are very spread out. Closeness approaching $\infty$ means the scores are effectively always tied. A closeness of $C$ means that a majority of the score differences will fall between $-1/C$ and $1/C$. If a game can only have a score difference of -1 or 1, its closeness will be exactly 1, no matter if it is biased or unbiased. If tie scores are also allowed, i.e. score differences of -1, 0, or 1, then we would expect the game’s closeness to be higher – and indeed this is the case, as closeness will always be $> 1$.

### 3 Rolling Sorted or Unsorted

Many games ask the players to roll a handful of dice. A method to assign the dice into pairs is required. In Risk, the dice are sorted from largest value rolled to smallest, which is the approach taken here. We also consider games in which the dice are rolled one at a time (or one die is rolled several times) and left unsorted. We now show how these two methods significantly change the distribution of score differences.

#### 3.1 Sorting Dice, with Ties

We first look at the case where each player rolls all $n$ of their $k$-sided dice and then sorts them in decreasing order. The two sets of dice are then paired and compared. If a player rolls more than one copy of the same number, the relative order of those two dice does not matter.

Figure 2 shows the distribution of score differences when each player rolls $n = 5$ dice and sorts them. We vary $k$, the number of sides. Ties are allowed, with neither player earning a point. We see the games all have a win bias of 0, as expected from the symmetry formed by players having the same rules. Tie percentage decreases as we increase the number of sides, as expected, since the more possible numbers to roll, the less likely the players will roll the same values. Increasing sides also decreases closeness, making higher score differences more likely to occur. For the case of $5d8$ and $5d10$, every possible score difference is approximately equally likely: large score differences are about as likely as close scores.
5d2 stands out as having a bell-shaped curve with significantly higher closeness: close games are more likely, but ties are also more likely as well. Nonetheless, 2-sided dice (e.g. coins) are not typically used in games, partly because standard coins are difficult to toss and keep from rolling off the table (see Coin Age and Shift for notable counter-examples, and some countries use square coins). However, stick dice – elongated dice that only land on the two long sides – do not roll away, and might be interesting items for game designers to investigate.

Figure 3 shows how changing the number of 6-sided dice affects the distribution of score differences. They remain symmetric with a win bias of 0, and after 2d6, adding more dice decreases the tie percentage. Closeness decreases as more dice are added, which makes sense as larger score differences are possible with more dice. 1d6 has a closeness greater than 1, because it allows 0-0 ties as well as games that end 1-0 or 0-1; without any ties, it would be exactly a closeness of 1.

3.2 Dice Unsorted, with Ties

We now examine the case where the dice are rolled and left unsorted. The dice could be rolled one at a time, possibly bringing out more drama as the battle is played out in individual dice pairings. Both players still roll n dice, but pair them in the order in which they were rolled rather than sorted by value, as shown in Figure 4. The player with the higher value earns a point, and if tied then neither player earns a point.

Although we will think of the dice being rolled one at a time (and actually generate them in our simulations this way), it is also possible for players to roll a handful of dice to quickly create a sequence, as shown in Figure 4b. A player first rolls a handful of dice on the table. The dice are then put in order from left to right as they settled on the table. If two dice have the same horizontal position on the table (as the 3 and 2 do in the example), the die further away from the player will come before the die that is near.

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3https://boardgamegeek.com/boardgame/146130/coin-age
4https://laboratory.vg/shift/
5One reviewer mentioned that their preferred method for rolling unsorted nd6 is to throw them against the inside of a sloped box lid: the dice line up in a random order as they slide to rest, and can be jiggled as needed.
In Figure 5, we examine how changing the number of sides of dice changes the distribution of ties and close games. We compare 2-sided dice (coins), 4-sided, 6-sided, 8-sided, and 10-sided dice. In all cases, the game is balanced, because the win bias is 0. With more sides, ties are less likely and closeness is generally lower. More sides therefore increase the odds of a lopsided victory with greater score differences between the players.

In Figure 6, we examine how changing the number of dice rolled affects the score difference. All of these games are balanced, since the win bias remains 0 in all cases.

3.3 Sorted versus Unsorted

In Figure 7, we review the effect of changing the way that dice are rolled, while keeping the same number of dice and number of sides ($5d6$ in this case). Rolling sorted has a flat distribution that leads to a higher likelihood of larger score differences, while rolling unsorted has a more normal-like distribution where closer games are more likely and closeness is higher. Higher closeness also increases tie percentage.

Tied games are much more common when rolling an even number of dice. Comparing with Figure 3, we see that rolling unsorted increases the tie percentage. With more sides, closeness is lower, as we saw with rolling sorted.

Game designers can choose the method appropriate for their games. In addition to choosing between rolling sorted or unsorted, the number of dice and number of dice sides can be changed. Using fewer sides on the dice increases closeness and tie percentage. Using fewer dice increases closeness and also generally increases the tie percentage. We address ties in the next sections.

4 Resolving Tied Battles

In the previous section, dice pairs tied in value did not give a point to either player. This leads to some situations in which the players get a 0 score difference and the overall game is tied (with as much as 24.6% for the $5d2$ case). For games in which $n$ is even, a score difference of 0 can occur (becoming less likely as $k$ increases).

A game designer might want to make such tied games impossible. One simple way would be to have Player A automatically win whenever...
the battle ends with a score difference of 0 – however this would have a massive bias in favour of Player A. In the above example, this would add an additional 24.6% bias which is likely unacceptable when trying to make the games close. To eliminate the bias over repeated battles, Player A and B could take turns receiving the win (perhaps by using a two-sided disk to indicate who will next receive the tiebreak).

Another simple way that would not have bias would be for the players to flip a coin (or some other random 50% chance event) to decide who is the winner of the battle. Using dice, the players could roll \(1d_k\) and let the player with the higher value win the battle. If they tie again, they repeat the \(1d_k\) roll until there is not a tie – we analyse this type of re-rolling in Section 8.

In the next few sections, we will examine some rule changes that make score differences of 0 impossible when \(n\) is odd. When \(n\) is even, score differences of 0 can still occur, and one of the above final tiebreaker methods can be used.

5 Favouring a Player

We now investigate breaking tied dice pairs by always having one player win a point when two dice are equal. We examine the case in which Player A will always win the point (as in Risk, in which defenders always win ties against attackers), but in general the same results apply if A and B are swapped. Favouring one player causes a bias, helping that player win more battles, so we examine ways to address this bias below.

5.1 Rolling Sorted, A Wins Ties

Figure 8 shows the score distributions when tied dice give a point to Player A. These distributions are clearly not symmetric, and are heavily skewed towards Player A, as reflected by the positive win bias. As one would expect, giving the ties to Player A causes that player to have an advantage over B. Increasing the number of sides on the die decreases the win bias; this is expected, as with more sides on the dice, the less likely it is for the players to both roll the same number. When \(n\) is odd, we also see that even score differences are no longer possible, and, most importantly, a tied score difference of 0 is no longer possible, so tie percentage is always 0. For the first time, we see an example of closeness increasing as the number of sides increases, because the distributions are less skewed towards large 5-0 lopsided wins.

### Table: Rolling Sorted, A Wins Ties

<table>
<thead>
<tr>
<th>Game</th>
<th>Win Bias</th>
<th>Tie %</th>
<th>Closeness</th>
</tr>
</thead>
<tbody>
<tr>
<td>5d2</td>
<td>89.06</td>
<td>0.00</td>
<td>0.238</td>
</tr>
<tr>
<td>5d4</td>
<td>54.79</td>
<td>0.00</td>
<td>0.271</td>
</tr>
<tr>
<td>5d6</td>
<td>38.21</td>
<td>0.00</td>
<td>0.282</td>
</tr>
<tr>
<td>5d8</td>
<td>29.13</td>
<td>0.00</td>
<td>0.287</td>
</tr>
<tr>
<td>5d10</td>
<td>23.48</td>
<td>0.00</td>
<td>0.289</td>
</tr>
</tbody>
</table>

Figure 8. Rolling \(k\)-sided dice sorted, A wins ties.

5.2 Rolling Unsorted, A Wins Ties

By switching to rolling dice unsorted, the closeness is increased for all numbers of dice, and the distribution is more centred, but there is still a significant bias towards Player A, as we can see from Figure 9. This is an improvement, but one might desire another way to reduce the bias.

### Table: Rolling Unsorted, A Wins Ties

<table>
<thead>
<tr>
<th>Game</th>
<th>Win Bias</th>
<th>Tie %</th>
<th>Closeness</th>
</tr>
</thead>
<tbody>
<tr>
<td>5d2</td>
<td>79.30</td>
<td>0.00</td>
<td>0.316</td>
</tr>
<tr>
<td>5d4</td>
<td>44.96</td>
<td>0.00</td>
<td>0.400</td>
</tr>
<tr>
<td>5d6</td>
<td>30.68</td>
<td>0.00</td>
<td>0.424</td>
</tr>
<tr>
<td>5d8</td>
<td>23.19</td>
<td>0.00</td>
<td>0.434</td>
</tr>
<tr>
<td>5d10</td>
<td>18.63</td>
<td>0.00</td>
<td>0.439</td>
</tr>
</tbody>
</table>

Figure 9. \(k\)-sided dice unsorted, A wins ties.

In conclusion, resolving tied pairs in favour of one player eliminates ties, but creates a large win bias. However, this can be reduced with more sides on the dice. This bias occurs for both rolling sorted and unsorted, although rolling unsorted
results in higher closeness and lower win bias. We now consider other ways to reduce this bias.

6 Reducing Bias with Fewer Dice

The bias introduced by having one player win ties can be undesirable for some designers and players, so we now look at a method of reducing this bias by having Player A roll fewer dice than Player B, to make up for the advantage they earn by winning ties. This method is used in Risk, in which the tie-winning bias towards Player A (defender) is reduced by allowing Player B (attacker) to roll an extra die when both sides are fighting with large armies. When rolling sorted, the dice are sorted in decreasing order, and the lowest-valued unmatched dice are ignored. When rolled unsorted, if one player rolls fewer dice, then there is no way to decide which should be ignored. We therefore only examine the case of rolling sorted.

Figure 10 shows the effect of requiring Player A to roll fewer dice. Rolling two or three fewer dice significantly favours Player B, and rolling the same number of dice favours Player A. However, Player A rolling 4d6 against Player B rolling 5d6 has a relatively balanced distribution, no longer significantly favouring one player over the other.

![Figure 10. A rolls fewer dice to control bias.](image)

However, ties once again occur for 4d6 versus 5d6 – they are possible in any battle in which Player A rolls an even number of dice – with a significant tie percentage (20.4).

Since having one fewer die made Player A and Player B relatively balanced when B rolls five dice, we can look at more cases when Player B rolls n dice. Figure 11 shows more cases in which Player A has one fewer die than Player B. Most of these cases are relatively balanced, although 1d6 versus 2d6 still gives a significant advantage to Player B. Note that the cases of 1d6 versus 2d6 and 2d6 versus 3d6 are used in Risk.

![Figure 11. Rolling one fewer die to control bias.](image)

Having Player A roll fewer dice reduces the bias introduced by having them win all ties. Rolling one fewer dice is the best choice that leads to the smallest win bias, and having both players roll more dice also reduces the win bias, but decreases the closeness. Instead of having the players rolling different numbers of dice, we next examine having the players roll dice with different numbers of sides.

7 Reducing Bias with Mixed Dice

Another way to reduce Player A’s tie-winning bias is to give Player B some dice with more sides. For example, we could have Player A roll five 6-sided dice and have Player B roll three 6-sided dice and two 8-sided dice, to give them a small advantage to help eliminate the advantage A receives for winning ties. Because bias does not occur when we allow ties, we will only examine using mixed dice for games in which Player A wins ties.

7.1 Mixed Dice Sorted, A Wins Ties

Figure 12 shows the distribution of score differences for mixes of d6 and d8 for Player B, while Player A always rolls 5d6. Adding more d8 adjusts the bias in favour of Player B, but adding too many creates a strong bias for Player B.
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<table>
<thead>
<tr>
<th>Game</th>
<th>win bias</th>
<th>tie %</th>
<th>closeness</th>
</tr>
</thead>
<tbody>
<tr>
<td>5d6/0d8</td>
<td>38.21</td>
<td>0.00</td>
<td>0.282</td>
</tr>
<tr>
<td>4d6/1d8</td>
<td>24.36</td>
<td>0.00</td>
<td>0.294</td>
</tr>
<tr>
<td>3d6/2d8</td>
<td>10.80</td>
<td>0.00</td>
<td>0.302</td>
</tr>
<tr>
<td>2d6/3d8</td>
<td>-2.24</td>
<td>0.00</td>
<td>0.305</td>
</tr>
<tr>
<td>1d6/4d8</td>
<td>-14.56</td>
<td>0.00</td>
<td>0.305</td>
</tr>
<tr>
<td>0d6/5d8</td>
<td>-25.98</td>
<td>0.00</td>
<td>0.301</td>
</tr>
</tbody>
</table>

Figure 12. Mixing d6 and d8 to control bias.

The fairest of these is when Player B rolls 2d6 and 3d8 against Player A’s 5d6 (drawn as a solid line in the figure), with a win bias of -2.24%. We analysed all possible mixes of five dice made of 6-sided, 8-sided, and 10-sided dice, and found that only three cases have a win minus loss bias under 10%, as shown in Figure 13.

The bias is still most balanced when Player B rolls 2d6 and 3d8 against Player A’s 5d6. However, by rolling 3d6/1d8/1d10, we can get a slight bias towards Player A, if that is desired.

<table>
<thead>
<tr>
<th>Game</th>
<th>win bias</th>
<th>tie %</th>
<th>closeness</th>
</tr>
</thead>
<tbody>
<tr>
<td>5d6/0d8</td>
<td>30.68</td>
<td>0.00</td>
<td>0.424</td>
</tr>
<tr>
<td>4d6/1d8</td>
<td>20.34</td>
<td>0.00</td>
<td>0.440</td>
</tr>
<tr>
<td>3d6/2d8</td>
<td>9.48</td>
<td>0.00</td>
<td>0.450</td>
</tr>
<tr>
<td>2d6/3d8</td>
<td>-1.61</td>
<td>0.00</td>
<td>0.452</td>
</tr>
<tr>
<td>1d6/4d8</td>
<td>-12.60</td>
<td>0.00</td>
<td>0.446</td>
</tr>
<tr>
<td>0d6/5d8</td>
<td>-23.19</td>
<td>0.00</td>
<td>0.434</td>
</tr>
</tbody>
</table>

Figure 14. Mixing d6 and d8 to control bias.

7.2 Mixed Dice Unsorted, A Wins Ties

We can do the same type of experiment for all variations of Player B rolling unsorted a mix of five d6s and d8s against Player A’s 5d6, getting the results as shown in Figure 14. By using 2d6 and 3d8, we can reduce the bias down to a small 1.61% in favour of Player B.
8 Re-Rolling Tied Dice

We now examine re-rolling tied dice as a final way to deal with tied pairs of dice. For example, it is common for players to roll a 1d6 at the start of a game to decide who moves first, and to re-roll the die in case of a tie. While this can be generalised to \( n \text{d} k \), it is cumbersome; this section explains why we believe that this is inadvisable in practice. Re-rolling can go on for many iterations, so we use Monte Carlo simulation to evaluate the probabilities empirically instead of exactly, since these games can theoretically continue indefinitely, with increasingly unlikely probability. We used \( N = 6^{10} = 60,466,176 \) simulations per game, as this is the same number used in our previous cases (see Appendix for this calculation). We use the \( \approx \) symbol to denote empirical rather than exact values in the figures.

8.1 Rolling Sorted, Re-Rolling Tied Dice

We first examine the case in which we roll a handful of dice and then sort them from highest to lowest. Untied dice pairs are scored, then tied dice pairs are re-rolled in sub-games until there are no more ties. Each player’s score from their initial roll, and all subsequent re-rolls, are summed together to give their overall score. The resulting score difference distributions for 5dk are shown in Figure 16. The battles are all unbiased, symmetrical, and without ties.

For 5d4, 5d6, 5d8 and 5d10, the distributions are effectively flat with low closeness and have approximately the same shape as when rolling sorted with ties (as in Figure 2) but now do not permit tie games. Compared to 5d4 and higher, 5d2 has a higher closeness. However, this closeness comes at the significant cost of requiring many re-rolls, as demonstrated in Figure 17. This shows that having more sides decreases the probability of a re-roll, and with 5d2 or 5d4 it is likely that players will have to make two or more re-rolls in practice, which is undesirable. Higher-sided\(^6\) dice are less likely to tie, so the probability of re-rolling decreases quickly when using six or more sides.

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\(^6\)That is, dice with a greater number of sides.
We also examine the effect of changing the number of dice while holding the number of sides fixed at 6 in Figure 18. The distributions are all flat, but closeness can be increased by using fewer dice, as we have seen in previous sections. The probability of re-rolls is also affected by the number of dice, as shown in Figure 19. For 1d6 and 2d6, the most common outcome is no re-rolls. Increasing the number of d6 makes re-rolling more likely, but in any case the probability of having to make many iterative re-rolls decreases rapidly (in contrast with d2).

Figure 19. Re-roll probability for Figure 18.

8.2 Rolling Unsorted, Re-Rolling Ties

Finally, we examine the case of rolling n k-sided dice unsorted when re-rolling tied dice pairs. The dice are rolled one at a time, and any time there is a tie, the two dice must be re-rolled until they are no longer tied. This occurs for each of the n dice. In practice, this would be tedious for players, but we present the analysis for completeness.

Because neither player is favoured, the metrics can be analytically calculated from the binomial distribution \( \binom{n}{w} p^w (1 - p)^{n-w} \), with \( w \) being the number of wins for Player A in the battle, \( n \) dice rolled, and probability \( p = .5 \) (no matter the value of \( k \)) of Player A winning each point. Given a score difference \( d \), we can calculate \( w = (n + d)/2 \).
We now consider a hybrid case, in which Player A wins all ties but must re-roll when they roll the die’s highest value (e.g. a 6 on a 6-sided die). This effectively means that Player A rolls \((k-1)\)-sided dice while Player B rolls \(k\)-sided dice. This gives an advantage to Player B to compensate for Player A’s tie-winning advantage.

Interestingly, this has the same effect as in the previous re-roll sections, for both rolling sorted or unsorted. Therefore, the plots are the same as in Figures 16, 18, 20, and 22. The benefit is that there is no re-rolling of ties. Only Player A has to re-roll when they roll \(k\) (unless they have access to \((k-1)\)-sided dice).

We can show analytically why this is unbiased for the simple case of one \(k\)-sided die. Player A re-rolls a \(k\) result, which is the same as rolling a \((k-1)\)-sided die. If Player A rolls a value of \(i\) with probability \(1/(k-1)\), then they win when Player B rolls a value \(\leq i\) with probability \(i/k\), since A wins ties. Calculating the expected number of wins for Player A, over all values of \(i\) from 1 to \(k-1\) we have:

\[
\sum_{i=1}^{k-1} \frac{i}{k(k-1)} = \frac{1}{2}
\]

which is independent of \(k\) and is always 1/2.

We have shown that re-rolling tied pairs gives unbiased results, but at the cost of requiring the players to re-roll, which can take longer. The expected number of re-rolls decreases with higher-sided dice. But we have shown other effective methods of eliminating ties which do not require re-rolls, so we suggest using these alternatives.

9 Risk & Risk 2210AD

We can use these results to examine how the original Risk compares with the popular variant Risk 2210AD [18]. In both games, the players roll sorted dice and the defender wins tied dice. As shown in Section 5.1, this gives a strong advantage to the defender when rolling the same number of dice, which can lead to a static game in which neither player wants to attack. To counteract this, both games allow the attacker to roll an extra die (3d6 versus 2d6). As shown in Section 6, this flips the advantage towards the attacker. An attacking advantage encourages players to play more aggressively.

In Risk 2210AD, special units called commanders and space stations replace one or more \(d6\) with \(d8\) in battles. As shown in Section 7, mixed dice bias the win rate towards the player rolling higher-valued dice. An attacker can use this to decrease the closeness and increase the predictability of battles, and a defender can use this to compensate for rolling fewer dice.

The rules in dice games require careful balancing, as the exact number of dice and number of sides can often have a large impact on the statistical outcome of the battles, as we have shown. Risk and Risk 2210AD are no exception, and their dice mechanisms appear to have been carefully tuned to provide reasonable win bias and closeness values.

10 Conclusion

We have demonstrated the use of win bias, tie percentage, and closeness to analyse a collection of dice battle variants for use as a component in a larger game. We introduce closeness, which is related to the precision statistic about 0, and matches the intuitive concept of a game being close. We have not seen this statistic used before.
to analyse games. The results of the previous sections let us make some general statements about this category of dice battles in which the number values are compared.

In Section 3 we showed that when tied pairs are allowed, rolling dice unsorted will result in higher closeness, i.e. a lower chance of games with large point differences; however, this comes at the cost of increasing the tie percentage. Using dice with fewer sides increases closeness, but also increases the tie percentage. Using fewer dice also increases closeness and generally increases the tie percentage.

Battles that end tied with a score difference of 0 can be broken with a coin flip or other 50/50 random event, as discussed in Section 4. However, we also explored rule changes that would cause odd-numbers of dice to never end in a tie. Breaking tied dice pairs in favour of one player, as shown in Section 5 eliminates ties but creates a large win bias, although this bias can be reduced by using higher-sided dice. This bias occurs for both rolling sorted and unsorted, although rolling unsorted gives higher closeness and slightly lower win bias.

In Section 6 we further reduced the win bias by having the favoured player roll fewer dice. Rolling one fewer die is the best choice that leads to the win bias closest to 0, and having both players roll more dice also makes the win bias closer to 0 but also decreases the closeness.

In Section 7 we reduced the win bias by having the unfavoured player roll different sided dice. Looking at all mixes of five dice composed of d6, d8 and d10, rolling 5d6 against 2d6/3d8 produced the smallest win bias, for both rolling sorted and unsorted. However, there was no way to completely eliminate the win bias.

In Section 8 we examined breaking tied pairs by re-rolling them. This gives unbiased results, but at the cost of a potentially lengthy re-rolling process. Using higher-sided dice or fewer dice reduces the expected number of re-rolls that will occur, but we recommend other tie-breaking methods that are less cumbersome for the players.

One surprising outcome of this study is that nd2 sorted with ties may be an under-used dice mechanic for games. This has high closeness and can be implemented by coins or stick dice, whose flat sides obviate some practical problems of round coins such as rolling off the table.

This article focussed on comparing dice values, but we are also doing a similar study of hit-based dice games, which includes analysing the effect of critical hits, following the same framework presented here.

For finer-grained control over the game experience, the designer can instead use a bag of dice tokens (e.g. small cardboard chits with dice faces printed on them) or a deck of dice cards to enforce that certain distributions are obeyed with local representation. This is choosing without replacement instead of choosing with replacement, which typically occurs in dice games. We are currently experimenting with examining similar games that use bags of dice tokens, using an exhaustive analysis similar to that done here.

In summary, there is no single best dice battle mechanism, and the designer must make a series of tradeoffs. We hope that this paper can provide some quantitative guidance to designers in search of dice games that exhibit particular properties or present a certain feel to the players. For designers wishing to use rules that we did not discuss in this paper, we hope it would not be difficult to use the techniques described above to evaluate how the players might experience the distribution of score differences by measuring win biases, ties and closeness.

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References

Appendix

In this appendix, we give analytical results for the probabilities and number of possible outcomes for many of the games studied in this paper. A more complete coverage of these probabilities and combinatorics can be found in [17].

A fair $k$-sided die has equal probability of rolling each of its $k$ sides, so the probability of rolling any particular number is $1/k$. Therefore, the total probability of rolling a value $v$ or less is $\sum_{v=1}^{\infty} 1/k = v/k$.

If we roll $n$ dice unsorted, there are $k^n$ different ways to roll the dice. Each way of rolling the dice, since the order matters, has an equal $k^{-n}$ chance. For example, if we roll five 6-sided dice unsorted, there are $6^5 = 7,776$ possible outcomes each with $1/7,776$ probability. If Player A is rolling a dice, and player B is rolling $b$ dice, then there are $k^a k^b = k^{a+b}$ possible outcomes. So, if each side rolls five 6-sided dice unsorted, there are $6^{10} = 60,466,176$ possible games that can occur, each equally likely. Rolling $5d10$ against $5d10$ has 10,000,000,000 different possible outcomes.

If we roll the $n$ dice sorted, then we can describe the probabilities using the multinomial distribution, a generalisation of the binomial distribution when there are $k$ possible outcomes for each trial. If one knows the outcome of a sorted roll had $x_i$ copies of $i$ (i.e. $x_1$ 1s, $x_2$ 2s, etc.), such that $x_1 + x_2 + ... + x_k = n$, the number of ways that particular outcome could have been rolled is:

$$\frac{n!}{x_1! x_2! ... x_k!}$$

The probability of rolling that outcome is:

$$\frac{n!}{x_1! x_2! ... x_k!} k^{-x_1 x_2 ... x_k}$$

For rolling $n$ $k$-sided dice sorted, the number of different possible results a player can roll is:

$$\binom{n + k - 1}{k - 1}$$

For example, for $5d6$, there are $\binom{5+6-1}{6-1} = \binom{10}{5} = 252$ unique ways to roll the dice, although
these are not of equal probability. For two players, there are $252^2 = 63,504$ ways to evaluate the game. This means that the rolling sorted calculations can be made much faster by only calculating each unique outcome once, but then multiplying the results by Equation 5 the number of ways each result can occur.

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**Shakashaka Challenges #1, #2 and #3**

Half-colour empty cells with triangles to create white squares and rectangles, as per the rules on p. 13.

**Challenge #1**

Challenge by ‘the axe and sword’ © Nikoli.

**Challenge #2**

Challenge by ‘cubic function’ © Nikoli.

**Challenge #3**

Challenge by Komeida © Nikoli.